

Comments on Unparticles

Benjamin Grinstein¹, Kenneth Intriligator¹ and Ira Z. Rothstein²

¹*Department of Physics, University of California, San Diego, La Jolla, CA 92093 USA*

²*Physics Department, Carnegie Mellon University, Pittsburgh PA 15213, USA*

We comment on several points concerning unparticles which have been overlooked in the literature. One regards Mack's unitarity constraint lower bounds on CFT operator dimensions, *e.g.*, $d_V \geq 3$ for primary, gauge invariant, vector unparticle operators. We correct the results in the literature to account for this, and also for a needed correction in the form of the propagator for vector and tensor unparticles. We show that the unitarity constraints can be directly related to unitarity requirements on scattering amplitudes of particles, *e.g.*, those of the standard model, coupled to the CFT operators. We also stress the existence of explicit standard model contact terms, which are generically induced by the coupling to the CFT (or any other hidden sector), and are subject to LEP bounds. Barring an unknown mechanism to tune away these contact interactions, they can swamp interference effects generated by the CFT. We illustrate these points in the context of a weakly coupled CFT example. A significant amount of the unparticle literature should be reconsidered or revised in light of the observations in this note.

1. Introduction

Observed, known particle physics is based on theories which have a mass gap and/or are free in the infrared. On the other hand, certain other theories – interacting conformal field theories – behave differently in the infra-red. Such theories have a traceless stress-energy tensor¹, with all coupling constants at fixed point values, $g_i = g_i^*$, where the beta functions vanish. Such theories do not have a traditional S-matrix description², because they do not have free, asymptotically separated in and out states. The possibility that certain gauge theories could have an interacting, non-Abelian Coulomb phase has a long history [5], *e.g.*, at weak coupling in theories which are barely asymptotically free [3]. In the context of supersymmetric theories, interacting conformal theories are also known to be quite common, and not limited to weak coupling. Over the years, there have been many proposed applications of renormalization group flows which approach near an interacting CFT in beyond the Standard Model model building, *e.g.*, walking technicolor.

A general class of extensions of the Standard Model involve coupling the visible sector to an otherwise hidden sector by some ultra-heavy fields of mass M . Integrating out the ultra-heavies induces higher dimension operators of the form

$$\mathcal{L} \supset \frac{c_{vh}}{M^{d_v+d_h-4}} \mathcal{O}_v \mathcal{O}_h + \frac{c_{vv}}{M^{2d_v-4}} \mathcal{O}_v \mathcal{O}_v + \frac{c_{vv,1}}{M^{2d_v-2}} \mathcal{O}_v \partial^2 \mathcal{O}_v + \dots, \quad (1.1)$$

where d_v is the operator dimension of the visible sector operator, d_h is that of the hidden sector operator, and c_{vh} and the other coefficients are dimensionless couplings. In recent work, Georgi [6] considered the possibility of such a coupling to an interacting CFT,

$$\mathcal{L} \supset \frac{\lambda}{M^{d_1+d_2-4}} \mathcal{O}_{SM} \mathcal{O}_{CFT}, \quad (1.2)$$

where λ is dimensionless, $d_1 = \Delta(\mathcal{O}_{SM})$ is the scaling dimension of the operator containing Standard Model fields and $d_2 = \Delta(\mathcal{O}_{CFT})$ is the dimension of the CFT operator, including its anomalous dimension, at the CFT renormalization group fixed point. The novel aspect

¹ In principle, a theory could be scale, but not conformal invariant, if T_μ^μ is a total divergence, rather than zero. In practice, however, scale invariant theories are quite generally also conformal [1] – the only known counterexamples are in $D = 2$ spacetime dimensions, and have other peculiarities: non-unitarity [2], or non-existence of operator 2-point functions [1]. In particular, the Banks-Zaks [3] type theories have, beyond scale invariance, symmetry under the full conformal group [1].

² This is how the conformal extension of the Poincare group evades the theorem of [4].

is the possibility of unusual d_2 values, and its effect on Standard Model scattering amplitudes. Some possible couplings, with both operators in (1.2) Lorentz scalars, vectors, and tensors, were discussed in [6]. Subsequent works explored other novel aspects of couplings to Lorentz vectors (especially the non-primary case $\mathcal{O}_{CFT}^\mu = \partial^\mu \mathcal{O}_{CFT}$) [7,8], scalar [9], spinor, and tensor CFT operators [10], and other interesting possible signatures [11,12].

The operators of a CFT are subject to general lower bounds on their scaling dimensions, from unitarity [13]: gauge invariant primary operators have scaling dimension

$$d \geq j_1 + j_2 + 2 - \delta_{j_1 j_2, 0}, \quad (1.3)$$

where (j_1, j_2) are the operator Lorentz spins. In particular, gauge invariant³ primary⁴ vector operators \mathcal{O}^μ have $d_V \geq 3$, with $d_V = 3$ if and only if the operator is a conserved current, $\partial_\mu \mathcal{O}^\mu = 0$. We review the constraints (1.3), and note that much of the unparticle literature has focused on d 's in a range which violate (1.3). (The latter remark also appears in [14].) The unitarity constraints were originally derived via a quite formal analysis. Below we provide a more physical description of how violations of unitarity arise in Standard Model scattering amplitudes, if they are coupled, as in (1.2), to unitarity violating operators.

We reconsider vector unparticles, with d in the range allowed by (1.3) and correct the form of the unparticle propagator for vector (and tensor) operators, to account for the fact that $\partial_\mu \mathcal{O}^\mu \neq 0$ if $d \neq 3$. We also discuss the situation for integer scaling dimension d , which leads to logs in momentum space and the necessity of a local counter-term.

As indicated in (1.1), coupling the Standard Model to another sector generally also induces Standard Model contact interactions $c_{vv} \neq 0$, which are subject to experimental bounds [15]. Most of the unparticle literature omits such contact interactions. Experimental constraints from effective contact interactions were first discussed in [11]. We here note that, in addition to the effective contact interaction associated with the c_{vh} term in (1.1), explicit contact interaction terms (the local operators associated with the c_{vv} terms in (1.1)) are generically also generated in the effective Lagrangian. Including these explicit contact terms can easily swamp interference effects arising from CFT interactions. Even if some unknown mechanism or fine tuning eliminates the explicit contact interactions at the scale M , they will generally still be generated at lower energy scales.

³ We stress that (1.3) applies only for gauge invariant operators. For example, a vector potential A_μ has $d = 1$, which does not contradict (1.3) because A_μ is not gauge invariant.

⁴ Primary means not a derivative of another operator. Since scalar operators have $d_S \geq 1$ (1.3), the non-primary vector operator $\partial_\mu \mathcal{O}$ has $d \geq 2$, as opposed to $d_V \geq 3$ for primary vectors.

2. Operator 2-point functions in CFT and unitarity

As mentioned in footnote 1, scale invariant theories are generally also conformally invariant. In this section, we discuss unitarity constraints on 4d CFTs, and in particular the result (1.3), which was originally obtained in the work [13]. Representations start with a *primary* operator, that is to say an operator which can not be written as $P_\mu = i\partial_\mu$ on another operator. The other operators in the representation are *descendants* of the primary operator, obtained by acting on the primary operator with P_μ . It is useful to focus on the primary operators, as results for descendants generally follow from acting with $P_\mu = i\partial_\mu$.

Conformal invariance completely determines the operator 2-point functions, in terms of their operator scaling dimensions, up to operator normalization constants. We quote expressions for primary operators. Two point functions of primary operators are only non-vanishing if the two operators have the same scaling dimension d (and of course also the same spins (j_1, j_2)). The two-point function of primary Lorentz scalar operators \mathcal{O} of dimension $d = \Delta(\mathcal{O})$ with their hermitian conjugates is

$$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = C_S \frac{1}{(2\pi)^2} \frac{1}{(x^2)^d}, \quad (2.1)$$

where C_S is a constant. For primary vector operators \mathcal{O}_μ , of dimension d , the 2-point functions are again determined, up to a normalization constant C_V [16]

$$\langle \mathcal{O}_\mu(x)^\dagger \mathcal{O}_\nu(0) \rangle = C_V \frac{1}{(2\pi)^2} \frac{I_{\mu\nu}(x)}{(x^2)^d}, \quad I_{\mu\nu} \equiv g_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}. \quad (2.2)$$

The particular form of $I_{\mu\nu}$ is completely determined by the conformal symmetry, in particular the special conformal transformation.⁵ For primary symmetric traceless, or anti-symmetric, 2-index tensor operators of dimension d , the two-point functions are [16],

$$\langle \mathcal{O}_{\mu\nu}(x)^\dagger \mathcal{O}_{\lambda\sigma}(0) \rangle = C_T \frac{1}{(2\pi)^2} \frac{(I_{\mu\lambda}(x)I_{\nu\sigma}(x) - \frac{1}{4}g_{\mu\nu}g_{\lambda\sigma}) \pm \mu \leftrightarrow \nu}{(x^2)^d}. \quad (2.3)$$

Similar expressions can be written for higher tensor representations, and spinor representations can be written in terms of Dirac γ^μ matrices.

⁵ It suffices to consider an infinitesimal special conformal transformation $x^\mu \rightarrow x'^\mu = x^\mu(1 + 2a \cdot x) - a^\mu x^2$. A scalar operator of dimension d transforms as $\mathcal{O}(x) \rightarrow \mathcal{O}'(x) = (1 + 2a \cdot x)^d \mathcal{O}(x')$, and a primary vector operator as $\mathcal{O}_\mu \rightarrow \mathcal{O}'_\mu = (1 + 2a \cdot x)^d (g_{\mu\nu} + 2a^\mu x^\nu - 2a^\nu x^\mu) \mathcal{O}_\nu(x')$. (Vector descendant operators, $\mathcal{O}_\mu = \partial_\mu \mathcal{O}$ transform differently.) Writing $\langle T \mathcal{O}_\mu(x) \mathcal{O}_\nu(0) \rangle = (A g_{\mu\nu} + B x_\mu x_\nu / x^2) / (x^2)^d$, invariance under the transformation requires $B = -2A$.

Unitarity requires that the constants appearing in the above be positive,

$$C > 0, \quad (2.4)$$

and the operators can then be rescaled to set the $C = 1$ if we like. The usual CFT argument for (2.4) follows from working with a Euclidean version of the theory, on $S^3 \times \mathbb{R}$, employing radial quantization, and mapping the operators to states via $\mathcal{O}(x \rightarrow 0)|0\rangle \rightarrow |\mathcal{O}\rangle$. Bras are obtained from the kets in this formalism by acting with inversions, so $\langle 0|\mathcal{O}(\infty)^\dagger \rightarrow \langle \mathcal{O}|$. It then follows that $C \propto |||\mathcal{O}||| > 0$. The unitarity conditions (2.4) lead to the unitarity conditions of [13] on the operator dimensions d . The idea is that even if a primary satisfies the positivity conditions (2.4), a descendant can violate it. This leads to the operator dimension requirement (1.3) of [13], to avoid having negative norm first descendants (acting with a single P_μ) for non-scalar operators, or second descendants (acting with $P_\mu P^\mu$) for scalar operators. See [17] for this approach and nice discussion of other relevant aspects.

To illustrate this, consider the scalar descendant $\mathcal{O}_{des} = P^\mu \mathcal{O}_\mu$ of a primary vector operator. Writing (2.2) as

$$\langle \mathcal{O}_\mu(x)^\dagger \mathcal{O}_\nu(0) \rangle = \frac{1}{(2\pi)^2} \frac{C_V}{2d} \left(\frac{1}{2(d-2)} g_{\mu\nu} \partial^2 - \frac{1}{d-1} \partial_\mu \partial_\nu \right) \frac{1}{(x^2)^{(d-1)}}, \quad (2.5)$$

we then have

$$\begin{aligned} \langle \mathcal{O}_{des}(x)^\dagger \mathcal{O}_{des}(0) \rangle &= \frac{1}{(2\pi)^2} C_V \frac{(d-3)}{4d(d-1)(d-2)} (\partial^2)^2 \frac{1}{(x^2)^{(d-1)}} \\ &= \frac{1}{(2\pi)^2} 4C_V (d-1)(d-3) \frac{1}{(x^2)^{(d+1)}}. \end{aligned} \quad (2.6)$$

This is of the general form (2.1) for the 2-point function of the scalar operator, $\partial^\mu \mathcal{O}_\mu$, but has the wrong sign if $1 < d < 3$. The range $d \leq 1$ can also be excluded; see Sec. 4. Note that it also follows from (2.6) that we can set $\partial^\mu \mathcal{O}_\mu = 0$ if, and only if the operator \mathcal{O}_μ has dimension $d = 3$, exactly. This fits with the fact that conserved currents have $d = 3$ exactly, with vanishing anomalous dimension.

3. 2-point functions in momentum space: “unparticle propagators.”

To consider the effect on SM scattering amplitudes of couplings (1.2) to a CFT, it is useful to Fourier transform the position space CFT 2-point functions, *e.g.*, (2.1), (2.2), and (2.3), into momentum space propagators. The Fourier integrals, and their inverses,

are generally singular, so continuations are required. First consider the scalar propagator. Evaluation of the integral in Euclidean space yields

$$\frac{1}{(2\pi)^2} \frac{1}{(x^2)^d} = \frac{\Gamma(2-d)}{4^{d-1}\Gamma(d)} \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} (k^2)^{d-2}. \quad (3.1)$$

Evaluation of the integral technically requires $0 < \text{Re}(d) < \frac{5}{4}$, but we assume it can be continued to all $\text{Re}(d) > 0$.

Similarly, for the position space form (2.2) for the two-point function of vector operators of operator dimension d , we find the momentum space propagator

$$\frac{1}{(2\pi)^2} \frac{g_{\mu\nu} - 2x_\mu x_\nu / x^2}{(x^2)^d} = \frac{(d-1)\Gamma(2-d)}{4^{d-1}\Gamma(d+1)} \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} (k^2)^{d-2} \left[g_{\mu\nu} - \frac{2(d-2)}{d-1} \frac{k_\mu k_\nu}{k^2} \right]. \quad (3.2)$$

Again, this computation was done in Euclidean space. To rotate to Minkowski we take $k^4 = -ik^0$ so $k^2 \rightarrow -k^2$ (metric is $(+ - - -)$). We need to keep the contour, rotated clockwise from the imaginary axis toward the real axis, off the poles or branch points on the real axis. These are at $k^0 = \pm |\vec{k}|$. So the contour goes under the real axis from $-\infty$ to $-|\vec{k}|$ and above the real axis from $|\vec{k}|$ to ∞ . As usual this can be summarized by an $i\epsilon$ prescription, that the poles/branch points are at $k^0 = \pm(|\vec{k}| - i\epsilon)$, or $k^2 + i\epsilon = 0$. So we can write, for the propagator of a dimension d vector in Minkowski space (with arbitrary normalization):

$$\int d^4x e^{-ik \cdot x} \langle 0 | T(O_\mu(x) O_\nu(0)) | 0 \rangle = -iC(-k^2 - i\epsilon)^{d-3} \left[k^2 g_{\mu\nu} - \frac{2(d-2)}{d-1} k_\mu k_\nu \right]. \quad (3.3)$$

This differs from the propagator of [6,7] in the relative size of the terms. The analogous expressions to (3.2) for tensor operators can be similarly written. For the traceless, symmetric tensor,

$$\begin{aligned} \frac{1}{(2\pi)^2} \frac{(I_{\mu\lambda}(x) I_{\nu\sigma}(x) + \mu \leftrightarrow \nu) - \frac{1}{2} g_{\mu\nu} g_{\lambda\sigma}}{(x^2)^d} = \\ \frac{\Gamma(2-d)}{4^{d-1}\Gamma(d+2)} \int \frac{d^4k}{(2\pi)^4} e^{ikx} (k^2)^{d-2} \left[d(d-1) (g_{\mu\lambda} g_{\nu\sigma} + \mu \leftrightarrow \nu) \right. \\ \left. + \frac{1}{2} [4 - d(d+1)] g_{\mu\nu} g_{\lambda\sigma} - 2(d-1)(d-2) \left(g_{\mu\lambda} \frac{k_\nu k_\sigma}{k^2} + g_{\mu\sigma} \frac{k_\nu k_\lambda}{k^2} + \mu \leftrightarrow \nu \right) \right. \\ \left. + 4(d-2) \left(g_{\mu\nu} \frac{k_\lambda k_\sigma}{k^2} + g_{\lambda\sigma} \frac{k_\mu k_\nu}{k^2} \right) + 8(d-2)(d-3) \frac{k_\mu k_\nu k_\lambda k_\sigma}{(k^2)^2} \right], \end{aligned} \quad (3.4)$$

and for the anti-symmetric tensor:

$$\begin{aligned} \frac{1}{(2\pi)^2} \frac{(I_{\mu\lambda}(x)I_{\nu\sigma}(x) - \mu \leftrightarrow \nu)}{(x^2)^d} = \\ - \frac{\Gamma(3-d)}{4^{d-1}\Gamma(d+1)} \int \frac{d^4k}{(2\pi)^4} e^{ikx} (k^2)^{d-2} \left[(g_{\mu\lambda}g_{\nu\sigma} - \mu \leftrightarrow \nu) \right. \\ \left. - 2 \left(g_{\mu\lambda} \frac{k_\nu k_\sigma}{k^2} + g_{\nu\sigma} \frac{k_\mu k_\lambda}{k^2} - \mu \leftrightarrow \nu \right) \right]. \end{aligned} \quad (3.5)$$

The propagators in (3.2), (3.4) and (3.5) have an apparent singularity when d is an integer. This is similar to that of the propagator of [6,7] (using $\Gamma(d)\Gamma(1-d) = \pi/\sin(\pi d)$). To illustrate what happens for integer d , consider the case of $d = 3$, where \mathcal{O}_μ is a conserved current. Taking $d \rightarrow 3$ in (3.2), there is a $1/(d-3)$ pole, which gives a local term, proportional to $\partial^2 \delta^{(4)}(x)$. Disregarding the local terms (there are additional local terms from the finite piece, which we also disregard), we have

$$\frac{1}{(2\pi)^2} \frac{g_{\mu\nu} - 2x_\mu x_\nu / x^2}{(x^2)^3} = \frac{1}{48} \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} (k^2 g_{\mu\nu} - k_\mu k_\nu) \ln(k^2). \quad (3.6)$$

As expected this has a log and is conserved. The scale of the log is irrelevant since it corresponds to a local term. If we write the scale in the log as μ , the μ dependence is cancelled by a contact counterterm $\frac{1}{24}(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) \delta^4(x)$ to the right hand side of (3.6).

The Fourier transforms (3.1)–(3.2) are only valid for $x \neq 0$. We have just seen that in the special case $d = 3$ an additional, local term is required to make the propagator independent of the scale μ . For non-integer d , the propagator is non-singular, and it is less clear if a local counter-term is still needed. In section 5 we will show that the correct propagator must include a contact counterterm even for non-integer d .

4. Unitarity conditions, re-obtained physically

In this section, we sketch Mack's derivation [13] of the constraints on irreducible representations of the conformal group, in particular for induced representations on Minkowski space. We then re-obtain the unitarity constraints on CFTs from requiring unitarity of scattering amplitudes of particles coupled to the CFT. While our results are not new, and in fact follow closely in the mathematics of Mack's original derivation, they give a new physical insight into the origin of the constraints.

Consider group transformations, $\phi \rightarrow U\phi$. If on this vector space we can find a group invariant, positive definite inner product (\cdot, \cdot) , then $U^\dagger U = 1$ on that space:

$$(\phi_2, \phi_1) = (U\phi_2, U\phi_1) = (\phi_2, U^\dagger U\phi_1).$$

For the conformal group, with primary operators ϕ , consider the inner product

$$(\phi_2, \phi_1) = \int d^4x \phi_2^\dagger(x) \Delta(x) \phi_1(x), \quad (4.1)$$

where $\Delta(x)$ is just the Wightman function

$$\Delta(x) = C \frac{P(x)}{(x^2 + i\epsilon x^0)^d}. \quad (4.2)$$

C is a non-zero constant, and $P(x)$ is the polynomial in $I_{\mu\nu}$ appropriate for the Lorentz spin of the representation, as in (2.1), (2.2), and (2.3). It is useful to consider the Fourier transformed version (where the ϵ in (4.2), amounting to $x_0 \rightarrow x_0 + i\epsilon$, leads to $\theta(k_0)$):

$$(\phi_2, \phi_1) = \int d^4k \tilde{\phi}_2^\dagger(-k) \tilde{\Delta}(k) \tilde{\phi}_1(k), \quad (4.3)$$

where $\tilde{\Delta}$ is the positive energy discontinuity across the cut of the momentum space propagator. By construction, this inner product is group invariant on primary fields. The representation is unitarity if (4.3) is moreover convergent and positive definite. As shown in [13], these conditions lead to the unitarity bounds (1.3) for (j_1, j_2) primary operators.

As we now discuss, these same conditions follow from the physical requirement of positivity of total scattering cross sections. Using the optical theorem, we relate the CFT unitarity requirements directly to the positivity of the imaginary part of the forward scattering amplitude, $\text{Im } A_{fwd} > 0$. This corresponds to the positivity of (4.3).

Consider first a scalar CFT operator $\mathcal{O}(x)$, coupled to an external source χ through a term in the lagrangian

$$\mathcal{L} \supset g\chi\mathcal{O} + \text{h.c.}$$

The source χ may, of course, create or annihilate any number of non-interacting (Standard Model) particles. The tree-level amplitude for $\chi \rightarrow \chi$, using (3.1) rotated to Minkowski space, is then

$$\mathcal{A} = g^2 C_S |\chi|^2 \frac{\Gamma(2-d)}{4^{d-1} \Gamma(d)} (-k^2 - i\epsilon)^{d-2},$$

where $|\chi|^2$ is from the creation and annihilation of external particles with total momentum k^μ . We are interested in the imaginary part in the forward region, $s = k^2 > 0$. We have dropped the t-channel contribution, because it has no imaginary part to leading order in $1/M^2$ (since $t < 0$). Using $\Gamma(1-x)\Gamma(x)\sin(\pi x) = \pi$, the imaginary part of the forward scattering amplitude is thus

$$\text{Im } \mathcal{A}_{\text{fwd}} = \frac{C_S \pi g^2 (d-1)}{4^{d-1} \Gamma(d)^2} |\chi|^2 \theta(k^0) \theta(k^2) (k^2)^{d-2}. \quad (4.4)$$

By the optical theorem this must be positive, which implies

$$C_S(d-1) \geq 0.$$

This, together with the condition that⁶ $C_S > 0$, shows that $d < 1$ is excluded.

The limit $d \rightarrow 1$ requires some additional care. Despite the $d-1$ factor, (4.4) does not identically vanish. Using $(d-1)\theta(k^2)/(k^2)^{2-d} \rightarrow \delta(k^2)$ as $d \rightarrow 1$ in Eq. (4.4) we have

$$\text{Im } \mathcal{A}_{\text{fwd}} = C_S \pi g^2 |\chi|^2 \theta(k^0) \delta(k^2) \quad \text{as } d \rightarrow 1 \quad (4.5)$$

(which is properly non-negative for $C_S > 0$). So $d = 1$ corresponds precisely to the exchange of a single scalar particle, with $k^2 = 0$, corresponding precisely to a free field.

Next consider a CFT vector operator, \mathcal{O}^μ of dimension d , which we couple to an external χ_μ via a term $\mathcal{L} \supset g \chi_\mu \mathcal{O}^\mu + \text{h.c.}$ Using (3.3) to compute the tree-level amplitude, this leads to a contribution with:

$$\text{Im } \mathcal{A}_{\text{fwd}} = -\frac{g^2 \pi C_V (d-1)^2}{4^{d-1} d \Gamma^2(d)} \left[\chi \cdot \chi^\dagger - \frac{2(d-2)}{d-1} \frac{|\chi \cdot k|^2}{k^2} \right] \theta(k^2) \theta(k^0) (k^2)^{d-2}. \quad (4.6)$$

Going to the center of mass frame $\vec{k} = 0$, (4.6) is positive if

$$\frac{C_V}{d} \left[|\vec{\chi}|^2 + \frac{d-3}{d-1} |\chi_0|^2 \right] \geq 0,$$

and since this condition must hold for arbitrary χ_μ , we have $C_V/d \geq 0$, and $(d-3)/(d-1) \geq 0$. This excludes $1 < d < 3$ and $d < 0$ since⁷ $C_V > 0$. Finally, let us exclude also $0 \leq d < 1$. We require the amplitude exists not just for in/out plane waves but also for arbitrary but

⁶ This was explained in the discussion below Eq. (2.4) but can also be seen directly from $\langle \overline{\mathcal{O}}^\dagger \mathcal{O} \rangle > 0$ with $\overline{\mathcal{O}} \equiv \int_{\mathcal{R}} d^4x \mathcal{O}$, where \mathcal{R} is a compact domain.

⁷ Again, this follows from $\langle \overline{\mathcal{O}}^\dagger \mathcal{O} \rangle > 0$ with $\overline{\mathcal{O}} \equiv \int_{\mathcal{R}} d^4x \mathcal{O}_\mu$, for any one fixed index μ .

nice in/out wave-functions. This means that we should be able to interpret $\chi_\mu(k)$ and $\chi_\nu^\dagger(-k)$ as independent functions and we require that the integral over k converges. The light-cone singularity of the integrand in

$$\int d^4k \phi_1(k) \phi_2(-k) \frac{1}{(k^2)^a} \quad (4.7)$$

is integrable for $a < 1$. In our case above the $(k^2)^{d-2}(k_\mu k_\nu/k^2)$ term gives the condition $3 - d < 1$ or $d > 2$. Combining the above conditions, we have shown that we must have $d \geq 3$ for vector operators \mathcal{O}^μ . Note that, unlike the scalar case, the vector amplitude (4.6) is smooth when the unitarity bound is saturated, $d \rightarrow 3$, and all $k^2 > 0$ contribute.

Whenever the unitarity bound inequalities (1.3) are saturated, the representation of the conformal group is smaller – some descendants are set to zero [13]. In the scalar case, when $d \rightarrow 1$, this is reflected in the conversion of $\theta(k^2) \rightarrow \delta(k^2)$ in (4.5). Indeed, (4.5) vanishes if $|\chi|^2 = k^2$, since $k^2\delta(k^2) = 0$; this corresponds to the fact that a scalar with $d = 1$ has a scalar second descendant, $\partial^2\mathcal{O}$, with vanishing norm, requiring setting $\partial^2\mathcal{O} = 0$. On the other hand, for the vector case (4.6), the limit where the unitarity bound is saturated, $d \rightarrow 3$, is smooth, and still involves $\theta(k^2)$ (rather than $\delta(k^2)$). When the unitarity bound is saturated for vectors, $d \rightarrow 3$, the operator with zero norm is $\partial_\mu\mathcal{O}^\mu = 0$, which corresponds to the vanishing of (4.6) for $\vec{\chi} = 0$ when $d = 3$.

The arguments above can be easily generalized for other tensors. Primaries with $j_1j_2 = 0$ are similar to the scalar case: when $d \rightarrow j_1 + j_2 + 1$, the $\theta(k^2)$ in $\text{Im } \mathcal{A}_{\text{fwd}}$ becomes $\delta(k^2)$, and some descendants are set to zero corresponding to $k^2\delta(k^2) = 0$. On the other hand, primaries with $j_1j_2 \neq 0$ are similar to the vector case: when $d \rightarrow j_1 + j_2 + 2$, the behavior of $\text{Im } \mathcal{A}_{\text{fwd}}$ is smooth, involving still $\theta(k^2)$ rather than $\delta(k^2)$; some first descendants have zero norm, and hence vanish, due to the tensor structure of the terms.

Consider the case of the anti-symmetric tensor, which is $(j_1, j_2) = (1, 0) + (0, 1)$ (self-dual and anti-self dual). The propagator, Eq. (3.5), has no d -dependence between the different tensor structures, but the overall coefficient of the absorptive part is, up to a positive quantity, $(d-2)(d-1)/d$. Convergence excludes $d < 1$ (as for vectors), so the unitary condition becomes $d \geq 2$. The limit $d \rightarrow 2$ requires care, because of the overall factor of $(d-2)$: of the terms in the tensor structure in (3.5), only those with inverse powers of k^2 survive, since they give a contribution as $d \rightarrow 2$ via $(d-2)\theta(k^2)/(k^2)^{3-d} \rightarrow \delta(k^2)$. For $d \rightarrow 2$, the $\delta(k^2)$ leads (since $k^2\delta(k^2) = 0$) to vanishing norm first descendants, which must be set to zero: $\epsilon^{\mu\nu\lambda\sigma} \partial_\nu \mathcal{O}_{\lambda\sigma} = \partial^\nu \mathcal{O}_{\nu\mu} = 0$. So $d = 2$ precisely corresponds to the exchange

of a free $U(1)$ field strength tensor, $\mathcal{O}_{\mu\nu} = F_{\mu\nu}$. Other $j_1 j_2 = 0$ primary operators are similar.

For the symmetric traceless tensor, $(j_1, j_2) = (1, 1)$, there are two differences from the $(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$ vector case above. First, convergence for wavepackets of the last term in Eq. (3.4) requires now $d > 3$. And second, the tensor structure has different d dependence. Coupling to a tensor source, $\chi_{\mu\nu}$ the condition from the χ_{00} or $\delta_{ij}\text{Tr}(\chi)$ components is $(d-4)(d-3) \geq 0$. Convergence and positivity lead to the condition $d \geq 4$. As in the case of the vector, the limit where the unitarity bound is saturated, $d \rightarrow 4$ is smooth. There is a vanishing norm first descendant (conservation law) $\partial^\mu O_{\mu\nu} = 0$, due to the tensor structure of the terms. Other $j_1 j_2 \neq 0$ primary operators are similar.

5. Weakly coupled CFT examples

5.1. Illustration of the unitarity inequalities

We here briefly demonstrate that the unitarity inequalities (1.3) are satisfied in a weakly coupled Banks-Zaks [3] type CFT. Start with an $SU(N_c)$ gauge theory with N_f flavors of Dirac fermions Q_f , $f = 1 \dots N_f$, in the fundamental representation. Take the limit of large N_c and large N_f , holding $\epsilon \equiv \frac{11}{2} - \frac{N_f}{N_c}$ fixed, with $0 < \epsilon$ so that the theory is asymptotically free, but just barely so, $\epsilon \ll 1$, and work to leading order in ϵ and $1/N_c$. The beta function for the $SU(N_c)$ gauge coupling g in this limit is $\beta(\alpha) = -b_1 \alpha^2 + b_2 \alpha^3$, with $\alpha = g^2/4\pi$ and $b_1 = N_c \epsilon/3\pi$ and $b_2 \approx 25N_c^2/16\pi^2$. There is a zero of $\beta(\alpha)$ at parametrically small 't Hooft coupling, $\alpha_* N_c \approx N_c b_1/b_2 = 16\pi\epsilon/75 \ll 1$, so we expect perturbation theory to be reliable, and work to lowest non-trivial order, $\mathcal{O}(\epsilon)$.

Consider first the scalar gauge invariant operators

$$\mathcal{O}_S = \delta^{fg} \overline{Q}_f Q_g. \quad (5.1)$$

To $\mathcal{O}(\epsilon)$, including the 1-loop anomalous dimension evaluated at the RG fixed point, these operators have $d_S = 3 - \frac{4\pi\epsilon}{25}$. Note that the anomalous dimension is negative, so $d_S < 3$. For small ϵ , these scalar operators remain well above⁸ the unitarity bound $d_S \geq 1$.

⁸ Indeed, the gap equations conjecture of Refs. [18-19] is that the theory is only conformal if $d_S \geq 2$, and below that instead has chiral symmetry breaking. To better test the unitarity bound for scalar operators, consider a simple modification of the above example: add a gauge singlet, scalar field ϕ , so $\mathcal{O} = \phi$ is a gauge invariant operator near the unitarity bound $d_S \geq 1$. To make ϕ interacting, include a term $\mathcal{L} \supset h\phi \overline{Q}_f Q_g$, and note that for $\alpha_* N_c = \mathcal{O}(\epsilon)$, where $\beta_g(g_*, h_*) = 0$, the Yukawa coupling also has a RG fixed point, $\beta_h(g_*, h_*) = 0$ at $h_*^2 = \mathcal{O}(\epsilon)$. As required by the unitarity bound, the Yukawa interaction indeed leads to ϕ having $\gamma_S \equiv d_S - 1 = +\mathcal{O}(\epsilon) > 0$.

Now consider the vector gauge invariant operators

$$J^\mu = C^{fg} \overline{Q}^f \gamma^\mu Q^g, \quad \mathcal{O}^\mu = D^{fg} \overline{Q}_f \gamma^\mu \gamma^5 Q_g, \quad (5.2)$$

where C^{fg} and D^{fg} are constants. Classically the operators (5.2) all have $d = 3$, so the unitarity bound $d_V \geq 3$ requires that their anomalous dimensions must be non-negative. The operators J^μ , for all C^{fg} , and \mathcal{O}^μ for the case of traceless D^{fg} are conserved currents, so they all have $d_V = 3$ exactly, without any quantum modification. On the other hand, the operator \mathcal{O}_μ , with $D^{fg} = \delta^{fg}$, is not conserved – it is anomalous, $\partial_\mu \mathcal{O}^\mu = -\frac{\alpha_* N_f}{2\pi} \Theta$, where $\Theta \equiv \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$. One can verify that the anomalous current indeed has $\gamma_A \equiv d_V - 3 > 0$, compatible with the unitarity bound. Writing the two-point function of the divergence of the anomalous current using the general form (2.6),

$$\langle \partial_\mu \mathcal{O}^\mu(x) \partial_\nu \mathcal{O}^\nu(0) \rangle = \frac{4C_V \gamma_A (2 + \gamma_A)}{(x^2)^{4+\gamma_A}} = \left(\frac{\alpha_* N_f}{2\pi} \right)^2 \langle \Theta(x) \Theta(0) \rangle, \quad (5.3)$$

with $C_V > 0$. Writing $\langle \Theta(x) \Theta(0) \rangle \approx C_S / (x^2)^4$ (working to lowest order in perturbation theory), one finds $C_S > 0$, and then (5.3) yields $\gamma_A \approx (\alpha_* N_f / 2\pi)^2 C_S / 8C_V$, so $\gamma_A > 0$, as expected. This evaluation of γ_A corresponds to the three-loop diagram formed from two copies of the 1-loop anomaly triangle, joined by their gluons in the internal loop.

5.2. Contact terms

In this section, we consider coupling a weakly coupled CFT [3] to the Standard Model, via a ultra-heavy messenger sector, say a massive vector exchange of ultra-heavy mass M_U that couples a SM current, j_μ , to a CFT vector operator, \mathcal{O}_μ . As a concrete example, suppose $j_\mu = \bar{e} \gamma^\mu e + \bar{\mu} \gamma^\mu \mu$. We do not assume that \mathcal{O}_μ is conserved, *e.g.*, \mathcal{O}_μ could be the anomalous current of the previous subsection. Integrating out the messenger at the scale M_U , this induces

$$\mathcal{L}_{\text{eff}} \supset D_1(M_U) \frac{1}{M_U^2} \mathcal{Q}_1, \quad \mathcal{Q}_1 \equiv j_\mu \mathcal{O}^\mu. \quad (5.4)$$

Integrating out the ultra-heavy will generally also induce the contact terms

$$\mathcal{L}_{\text{eff}} \supset D_2(M_U) \frac{1}{M_U^2} \mathcal{Q}_2 - D_2(M_U) \frac{1}{M_U^4} j_\mu \partial^2 j^\mu + \dots, \quad \mathcal{Q}_2 \equiv j_\mu j^\mu, \quad (5.5)$$

where the two terms have the same coefficients because they come from expanding the ultra-heavy propagator $(M_U^2 + \partial^2)^{-1} = 1/M_U^2 - \partial^2/M_U^4 + \dots$

In [6,7] and following works on unparticles, generally only the terms (5.4) are included, without accounting for the induced contact terms (5.5). Effective contact interactions associated with (5.4) were considered in [11]. The point of this section will be to illustrate that the explicit contact terms in (5.5) are also needed. In particular, as we will discuss, there is important mixing between the term (5.4) and the second term in (5.5) in, *e.g.*, $e^+e^- \rightarrow \mu^+\mu^-$ scattering. For exclusive scattering, the first term in (5.5) is a more important, leading order effect, which would in general overwhelm the apparent unparticle effects. Of course, such contact terms do not contribute directly to the peculiar phase space effects associated with unparticle production. But the cross section for direct unparticle production is $\sim 1/M_U^4$ while interference effects are $\sim 1/M_U^2$. Observation of unparticle effects would likely come after observation of these leading effects, except perhaps in models with a more complicated messenger sector in which D_1 , but not D_2 , is suppressed.

The amplitude we are interested in is

$$\langle \mu\mu | \mathcal{H}_{\text{eff}} | ee \rangle, \quad \mathcal{H}_{\text{eff}} = \frac{1}{M_U^4} \sum_{i=1,2} C_i \mathcal{O}_i, \quad (5.6)$$

where the dimension 8 operators are

$$\begin{aligned} \mathcal{O}_1(0) &\equiv \int d^4x T(\mathcal{Q}_1(x) \mathcal{Q}_1(0)), \\ \mathcal{O}_2(0) &\equiv -j^\mu(0) \partial^2 j_\mu(0). \end{aligned} \quad (5.7)$$

We have included \mathcal{O}_2 because exchange of the BZ particles between two insertions of \mathcal{O}_1 requires a $j^\mu j_\mu$ counterterm, as seen from the discussion after (3.6). There is also a contribution from \mathcal{Q}_2 but we postpone including this to the end of this analysis, since there are no subtleties associated with it.

Neglecting SM interactions (j_μ acts as a background field) the insertion of \mathcal{O}_1 is just the CFT vector propagator $\langle \mathcal{O}_\mu(k) \mathcal{O}_\nu(-k) \rangle$, given by (3.3), contracted with $j^\mu(k) j^\nu(-k)$. (Because we chose a conserved current for j^μ , only the $g_{\mu\nu}(-k^2 - i\epsilon)^{d-2}$ term of the propagator contributes). Denoting by Γ_i the amputated Green function with an insertion of \mathcal{O}_i , we have renormalization group equations

$$\left[\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right] \Gamma_i = -\gamma_{ij}(g) \Gamma_j$$

where $t = \ln \mu$ and γ is the anomalous dimension matrix for the operators $\mathcal{O}_{1,2}$. We are interested in $d-3 = \gamma_{11} \neq 0$, so \mathcal{O}_μ is not conserved. Since we neglect SM interactions,

$\gamma_{21} = \gamma_{22} = 0$, but $\gamma_{12} \neq 0$ and starts at order g^0 , *i.e.*, it is present in the free field theory case. Close to the IR fixed point, $g = g_*$, the solution is

$$\begin{aligned}\Gamma_2(s, \mu) &= s\hat{\Gamma}_2 \\ \Gamma_1(s, \mu) &= -\frac{\gamma_{12}(g_*)}{\gamma_{11}(g_*)}\Gamma_2 + \left[s\hat{\Gamma}_1 + \frac{\gamma_{12}(g_*)}{\gamma_{11}(g_*)}\Gamma_2 \right] \left(\frac{\sqrt{s}}{\mu} \right)^{\gamma_{11}(g_*)},\end{aligned}\tag{5.8}$$

where $\hat{\Gamma}_i$ are constants and $s = k^2$. The constant $\hat{\Gamma}_1$ is fixed by the normalization of the operator \mathcal{O}_μ . If we could set $\Gamma_2 = 0$ then $\Gamma_1(s, \mu)$ would precisely correspond to the propagator (3.2) with $d-3 = \gamma_{11}(g_*)/2$. But this is not possible. The Γ_2 terms in $\Gamma_1(s, \mu)$ in (5.8) are crucial in computing physical quantities consistently, and can not be set to zero. (While $\hat{\Gamma}_2$ is arbitrary, the product $\gamma_{12}(g_*)\hat{\Gamma}_2$ is independent of any normalization convention for \mathcal{O}_2 .) The contact term \mathcal{O}_2 must be included.

To see this we return to the computation of the amplitude in (5.6). We run \mathcal{H}_{eff} in (5.6) to low energies, determining the running of the coefficient's C_i by insisting that the amplitude be μ -independent:

$$\left[\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right] C_i = \gamma_{ji}(g) C_j.\tag{5.9}$$

We can run these equations from the far UV, down to the IR. There are matching conditions at M_U ,

$$C_1(M_U) = D_1^2(M_U) \quad C_2(M_U) = D_2(M_U).$$

The second of these is explained following (5.5). We do the RG running down in two steps. In step 1, we run from the far UV, where the BZ gauge coupling $g \approx 0$, down to the “dimensional transmutation scale,” Λ_U , where the BZ gauge coupling runs near the IR fixed point value, $g \approx g_*$, where $\beta(g_*) = 0$. In step 2, we integrate (5.9) assuming that $g \approx g_*$, so we take $\beta(g) \approx 0$ and $\gamma(g) \approx \gamma(g_*)$. In step 1, the Green functions and coefficient functions are nearly constant, since $g \approx 0$. The one exception is C_2 which runs even at $g = 0$. Setting $\gamma_{12}(g) \approx \gamma_{12}^{(0)} = \text{constant}$, C_2 runs down to

$$C_2(\Lambda_U) \approx C_2(M_U) + \gamma_{12}^{(0)} C_1 \ln(\Lambda_U/M_U).\tag{5.10}$$

In step 2 the solution to (5.9) is,

$$C_1(\mu) = \left(\frac{\mu}{\Lambda_U} \right)^{\gamma_{11}(g_*)} C_1(\Lambda_U), \quad C_2(\mu) = C_2(\Lambda_U) + \frac{\gamma_{12}(g_*)}{\gamma_{11}(g_*)} \left[\left(\frac{\mu}{\Lambda_U} \right)^{\gamma_{11}(g_*)} - 1 \right] C_1(\Lambda_U).\tag{5.11}$$

Upon combining (5.8) and (5.11), it is easy to see that the amplitude is explicitly μ independent, as expected. But had we ignored the contact term in the propagator (Γ_2 in the second equation in (5.8)) there would have been residual μ dependence. As mentioned following (3.6), a contact term is needed for $d_V = 3$. We have now shown that it is needed also for $d_V \neq 3$. The above results indeed recover the $d_V = 3$ case discussed after (3.6) upon taking the limit $\gamma_{11} \rightarrow 0$ (the logs in (5.10) and (5.11) combine to give simply a $\ln(k^2/M_U^2)$ in the propagator of Eq. (3.6)).

As mentioned at the start of this section, the contribution from the operator \mathcal{Q}_2 to the $ee \rightarrow \mu\mu$ amplitude is much bigger than the contributions of \mathcal{O}_i which we have been discussing. If E is the center of mass energy then the ratio of the the CFT exchange amplitudes to the contact one is roughly

$$\frac{\mathcal{A}_{\text{unparticle}}}{\mathcal{A}_{\text{contact}}} \sim \frac{D_2^2}{D_1} \left(\frac{E}{M_U} \right)^2 \left(\frac{E}{\Lambda_U} \right)^{2(d-3)}.$$

Since $E < \Lambda_U < M_U$ both E -dependent factors tend to suppress this ratio. Even if Λ_U were close to M_U , as the energy E is raised towards M_U the effect of the s -channel resonance becomes apparent and continues to overwhelm the CFT exchange effect. The strategy for discovery should begin by observation of the contact interactions, which would fix the scale M_U , followed by high precision measurements to detect the small residual unparticle effects. In models with scalar unparticle exchange the contact interaction may be less important, since d_S can be lower, $d_S \geq 1$.

6. Amplitudes for vector unparticles, corrected

Consider the decay $t \rightarrow q + U$ where $q = u$ or c and U stands for a vector “unparticle.” Following Ref. [6] we take the effective interaction

$$\frac{\lambda}{\Lambda^{d-1}} \bar{q} \gamma_\mu (1 - \gamma_5) t \mathcal{O}^\mu + \text{h.c.}$$

but we take \mathcal{O}^μ to be a unitary, primary operator ([6] takes a descendant \mathcal{O}^μ). The phase space is as in Ref. [6] (we neglect the mass of the final state quark) but the amplitude has a factor

$$-g_{\mu\nu} + a k_\mu k_\nu / k^2 \tag{6.1}$$

from the vector matrix element. Here k is the unparticle momentum and $a = 2(d - 2)/(d - 1)$. The interesting range is $d > 3$, where $a \neq 1$ (since $\partial_\mu \mathcal{O}^\mu \neq 0$). We'll keep a as a parameter, to compare easily with the incorrect result from using $a = 1$.

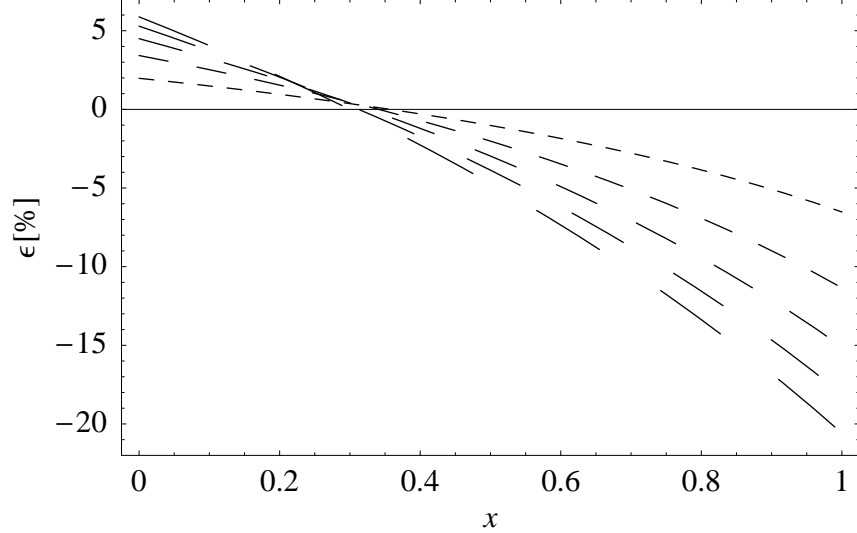


Figure 1: Fractional error ϵ in spectrum for the decay $t \rightarrow q + U$ as a function of $x = 2E/m_t$ when the coefficient a in (6.1) is set to 1 instead of its correct value $2(d - 2)/(d - 1)$. The plots correspond to $d = 3.2, 3.4, 3.6, 3.8, 4.0$ with the length of dashing increasing with d .

Using the notation of Ref. [6] for the phase space and its normalization factor we have

$$\frac{d\Gamma}{dx} = m_t \frac{A_d}{8\pi^2} \left(\frac{\lambda m_t^{d-1}}{\Lambda^{d-1}} \right)^2 x^2 (1-x)^{d-2} \left[1 - \frac{1}{2}a + \frac{(1 - \frac{1}{2}x)a}{1-x} \right],$$

where $x = 2E/m_t$. The spectrum is normalization independent, so it is perhaps more interesting:

$$\frac{1}{\Gamma} \frac{d\Gamma}{dx} = \frac{d(d^2 - 1)(d - 2)}{2d - 4 + a(d + 1)} x^2 (1-x)^{d-2} \left[1 - \frac{1}{2}a + \frac{(1 - \frac{1}{2}x)a}{1-x} \right].$$

In Fig.1 we show the fractional error

$$\epsilon \equiv \frac{\Delta \left(\frac{1}{\Gamma} \frac{d\Gamma}{dx} \right)}{\frac{1}{\Gamma} \frac{d\Gamma}{dx}}$$

where the difference is between the spectrum with the incorrect value, $a = 1$, and with the correct one, $a = 2(d - 2)/(d - 1)$.

7. Summary

We have commented on several points concerning CFTs⁹ which have been overlooked in the unparticle literature:

1. Unitarity imposes lower bounds on the dimensions of operators (there is no upper bound, nor problematic behavior for integer dimensions). In particular $d \geq 3$ for vectors and $d \geq 4$ for symmetric, traceless tensors.
2. Only when the unitarity bound on the dimension is saturated does the operator satisfy free field equations of motion or conservation laws. Correspondingly, the tensor structure of the propagators is modified.
3. Coupling the CFT to the SM via the exchange of an ultraheavy particle necessarily introduces SM contact interactions, which generally dominate over other unparticle interference effects. Moreover, CFT exchange induces additional SM contact interactions (which cure the apparent problems with integer dimensions).

A number of interesting ideas and possible effects have been considered in the unparticle literature. Where appropriate, the literature can be reanalyzed in light of the observations in this paper.

Acknowledgments We thank P. Puttayararat for discussions. The research of BG and KI is supported in part by Department of Energy under contract DOE-FG03-97ER40546 and that of IR by grants DOE-ER-40682-143 and DEAC02-6CH03000.

⁹ Again, scale invariant theories are quite generally also conformally invariant [1]. There are no known examples of 4d unitary theories which are scale but not conformally invariant, and it is quite possible that such theories cannot exist. Nevertheless, some of the unparticle literature misguidedly attempts to ignore the constraints of conformal invariance, by restricting their considerations to the hypothetical, perhaps non-existent, class of theories which are scale invariant but not conformal. For the sake of completeness, we note that there are lower bounds on d even without using conformal symmetry. The bound $d \geq 1$ for scalars, reviewed in section 4, did not use the conformal symmetry. More generally, even without imposing conformal symmetry, unitarity and convergence of (4.7) for the term in the propagator with maximum spin j (maximum number of k^2 's in the denominator) gives $d \geq j + 1$. And since $j_{max} = j_1 + j_2$, this gives $d \geq j_1 + j_2 + 1$. (Arbitrarily omitting the maximum j terms from the propagator would lead to a weaker inequality; *e.g.*, keeping only the $j = 0$ term, proportional to $g_{\mu\nu}$'s, would weaken the bound to $d \geq 1$.) The stronger constraint of conformal symmetry only has the effect of strengthening the bound for $j_1 j_2 \neq 0$, to $d \geq j_1 + j_2 + 2$.

References

- [1] J. Polchinski, “SCALE AND CONFORMAL INVARIANCE IN QUANTUM FIELD THEORY,” Nucl. Phys. B **303**, 226 (1988).
- [2] V. Riva and J. L. Cardy, “Scale and conformal invariance in field theory: A physical counterexample,” Phys. Lett. B **622**, 339 (2005) [arXiv:hep-th/0504197].
- [3] T. Banks and A. Zaks, “On The Phase Structure Of Vector - Like Gauge Theories With Massless Fermions,” Nucl. Phys. B **196**, 189 (1982).
- [4] S. R. Coleman and J. Mandula, “ALL POSSIBLE SYMMETRIES OF THE S MATRIX,” Phys. Rev. **159**, 1251 (1967).
- [5] D. J. Gross and F. Wilczek, “Asymptotically Free Gauge Theories. 2,” Phys. Rev. D **9**, 980 (1974).
- [6] H. Georgi, “Unparticle Physics,” Phys. Rev. Lett. **98**, 221601 (2007) [arXiv:hep-ph/0703260].
- [7] H. Georgi, “Another Odd Thing About Unparticle Physics,” Phys. Lett. B **650**, 275 (2007) [arXiv:0704.2457 [hep-ph]].
- [8] K. Cheung, W. Y. Keung and T. C. Yuan, “Collider signals of unparticle physics,” Phys. Rev. Lett. **99**, 051803 (2007) [arXiv:0704.2588 [hep-ph]].
- [9] P. J. Fox, A. Rajaraman and Y. Shirman, “Bounds on Unparticles from the Higgs Sector,” Phys. Rev. D **76**, 075004 (2007) [arXiv:0705.3092 [hep-ph]].
- [10] H. Goldberg and P. Nath, “Ungravity and Its Possible Test,” arXiv:0706.3898 [hep-ph].
- [11] M. Bander, J. L. Feng, A. Rajaraman and Y. Shirman, “Unparticles: Scales and High Energy Probes,” Phys. Rev. D **76**, 115002 (2007) [arXiv:0706.2677 [hep-ph]].
- [12] M. J. Strassler, “Why Unparticle Models with Mass Gaps are Examples of Hidden Valleys,” arXiv:0801.0629 [hep-ph].
- [13] G. Mack, “All Unitary Ray Representations Of The Conformal Group SU(2,2) With Positive Energy,” Commun. Math. Phys. **55**, 1 (1977).
- [14] Y. Nakayama, “SUSY Unparticle and Conformal Sequestering,” Phys. Rev. D **76**, 105009 (2007) [arXiv:0707.2451 [hep-ph]].
- [15] J. Alcaraz *et al.* [ALEPH Collaboration], “A combination of preliminary electroweak measurements and constraints on the standard model,” arXiv:hep-ex/0612034.
- [16] H. Osborn and A. C. Petkou, “Implications of conformal invariance in field theories for general dimensions,” Annals Phys. **231**, 311 (1994) [arXiv:hep-th/9307010].
- [17] S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field theories,” Adv. Theor. Math. Phys. **2**, 781 (1998) [arXiv:hep-th/9712074].
- [18] A. G. Cohen and H. Georgi, “Walking Beyond The Rainbow,” Nucl. Phys. B **314**, 7 (1989).
- [19] T. Appelquist, J. Terning and L. C. R. Wijewardhana, “The Zero Temperature Chiral Phase Transition in SU(N) Gauge Theories,” Phys. Rev. Lett. **77**, 1214 (1996) [arXiv:hep-ph/9602385].